

AN EXAMPLE WHERE TOPOLOGICAL ENTROPY IS CONTINUOUS

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ABSTRACT. Let ent denote topological entropy, and let $C^r(S^1, S^1)$ denote the space of continuous functions of the circle to itself having r continuous derivatives with the C^r (uniform) topology. Let f_0 denote a particular C^2 map of the circle (f_0 is the first bifurcation point one comes to in a bifurcation from a full three shift to a map with finite nonwandering set). The main results of this paper are the following:

THEOREM A. *The map $\text{ent}: C^0(S^1, S^1) \rightarrow \mathbb{R} \cup \{\infty\}$ is lower-semicontinuous at f_0 .*

THEOREM B. *The map $\text{ent}: C^2(S^1, S^1) \rightarrow \mathbb{R}$ is continuous at f_0 .*

In proving these two theorems several general results on entropy of mappings of the circle are proved.

1. Introduction and statement of results. This paper is concerned with topological entropy of mappings of the circle. After proving some general results we show that for an example of a map $f_0 \in C^2(S^1, S^1)$, topological entropy (considered as a map from $C^2(S^1, S^1)$ to the nonnegative real numbers) is continuous at f_0 . The example is nontrivial in the sense that entropy is not locally constant at f_0 . The main results will be stated precisely after a description of the example is given (see Theorems A, B, C, and D).

Let f_0 denote any map in $C^2(S^1, S^1)$ that satisfies the following conditions (see §2 for definitions and see Figure 1 for a picture):

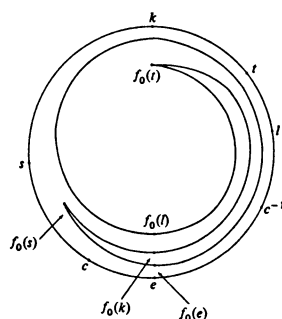


FIGURE 1

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(1) f_0 has an expanding fixed point e , and a contracting fixed point c , and f_0 is orientation preserving at e and c .

(2) f_0 has nondegenerate singularities t and s , and these are the only singularities of f_0 .

(3) There are points $l, k \in S^1$ with $f_0(l) = f_0(k) = e$.

(4) The points $e, l, t, k, s, f_0(s)$, and c are distinct, and are in order around the circle in the counter-clockwise direction.

(5) f_0 is one to one on each of the intervals (e, l) , (l, t) , (t, s) , (s, e) . Here (a, b) denotes the open arc from a counter-clockwise to b .

(6) (k, e) is in the stable manifold of c .

(7) $f_0(t) = k$.

The map f_0 is pictured in Figure 1. In Figure 1, we have drawn the image of f_0 inside S^1 , but of course f_0 maps S^1 onto itself.

Let $C^r(S^1, S^1)$ denote the space of continuous functions of the circle to itself having r continuous derivatives, with the C^r (uniform) topology. Let ent denote topological entropy (see §2 for definition). Our main results are the following.

THEOREM A. *The map $\text{ent}: C^0(S^1, S^1) \rightarrow R \cup \{\infty\}$ is lower-semicontinuous at f_0 .*

THEOREM B. *The map $\text{ent}: C^2(S^1, S^1) \rightarrow R$ is continuous at f_0 .*

In proving Theorems A and B we obtain some general results of independent interest on entropy of mappings of the circle. We discuss some of these in the outline of the paper which follows.

In §3 we prove the following theorem:

THEOREM C. *Let $f \in C^1(S^1, S^1)$. Suppose there is a subset W of S^1 , and a finite collection of proper closed intervals of S^1 , $\{I(1), I(2), \dots, I(p)\}$, which satisfy the following:*

(1) *Either $W = S^1$ or $W = S^1 - (O_1 \cup \dots \cup O_m)$ where O_i is a component of the stable manifold of a contracting periodic point C_i for $i = 1, \dots, m$.*

(2) $W = I(1) \cup \dots \cup I(p)$.

(3) *For $j = 1, \dots, p$, f maps $I(j)$ homeomorphically onto its image.*

(4) *For any $i = 1, \dots, p$ and $j = 1, \dots, p$, $f(I(i)) \cap I(j)$ is an interval.*

Then $\text{ent}(f) \leq \ln(p)$.

Note that given any C^1 map of the circle with finitely many singularities, from the basic picture of the map (i.e., a picture as in Figure 1) one can obtain (by using Theorem C) an upper bound for the entropy of the map. In particular, Theorem C implies that if g is near f_0 in $C^2(S^1, S^1)$ then $\text{ent}(g) \leq \ln(3)$ (see the proof of Theorem B in §7 for details).

Let $f \in C^1(S^1, S^1)$ and suppose that f has a singularity, and that all

singularities of f are in the stable manifolds of sinks. In §4 we obtain a matrix A associated to f (see Definition 6, the choice of A is not unique) and prove the following theorem.

THEOREM D. $\text{ent}(f) = \ln(\lambda(A))$ where $\lambda(A)$ is the largest eigenvalue of A (\ln denotes the natural logarithm).

In §5 we construct a sequence (f_n) of maps in $C^2(S^1, S^1)$, and we show (using Theorem D) that $\text{ent}(f_n)$ approaches $\ln(3)$ as n approaches infinity. Each f_n has the same basic picture as f_0 , but $f_n(t) \in (t, k)$.

In §6 we prove a comparison theorem, which we use in §7 to show that for n arbitrarily large, if g is close enough to f_0 in $C^0(S^1, S^1)$ then $\text{ent}(g) > \text{ent}(f_n)$. Combining these results we obtain Theorems A and B, along with the fact that $\text{ent}(f_0) = \ln(3)$. It is easy to see that entropy is not locally constant at f_0 (see Proposition 9 of §5 for details).

Topological entropy was defined by Adler, Konheim, and McAndrew in [1]. Other papers on the subject include [2], [3], [4], [6], and [7]. In [2], it is shown that entropy as a map on $C^0(S^1, S^1)$ is not continuous at any map with finite entropy, while, in [6], an example is given where entropy is not continuous in the diffeomorphism case (on a manifold of dimension three).

Continuity of entropy may be thought of as a kind of quantitative stability in the sense that for dynamical systems near the given one the “amount of action” will be close to the given one.

Note that if one perturbs the map f_0 to an Axiom A map g (with only one sink c), whose basic picture is similar to f_0 except that $g(t) \in (k, c)$, then $\Omega(g) - \{c\}$ will be equivalent to the full one-sided shift on three symbols (see §2 for definitions). So f_0 may be thought of as the first bifurcation point one comes to, in a bifurcation from the full shift on three symbols to a map with finite nonwandering set.

Based on the results of this paper we make the following conjecture.

CONJECTURE E. *If $f \in C^2(S^1, S^1)$ and all singularities of f are nondegenerate then topological entropy (as a map from $C^2(S^1, S^1)$ to the nonnegative real numbers) is continuous at f .*

We also pose the following questions.

(1) Is topological entropy continuous everywhere on $C^2(S^1, S^1)$, or even $C^1(S^1, S^1)$?

(2) If M is a compact manifold is the property, topological entropy is continuous at f , a generic property of $C^r(M, M)$ (for suitable r)?

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2. Preliminary definitions and results. Let $f \in C^1(S^1, S^1)$. A point $x \in S^1$ is called a periodic point of f if $f^n(x) = x$ for some positive integer n . The

minimum of $\{n: f^n(x) = x\}$ is called the period of x .

A periodic point x of period n is called a sink or contracting periodic point if $|Df^n(x)| < 1$. In this case we define the stable manifold of x , denoted $W^s(x)$, by

$$W^s(x) = \left\{ y \in S^1: \lim_{k \rightarrow \infty} f^{nk}(y) = x \right\}.$$

If $y \in W^s(x)$ we define the semilocal stable manifold of y (denoted $\text{slsm}(y)$) to be the component of $W^s(x)$ which contains y .

A periodic point x of period n is called expanding if $|Df^n(x)| > 1$.

A point $x \in S^1$ is called a singularity of f if $Df(x) = 0$. A singularity x is said to be nondegenerate if $D^2f(x) \neq 0$.

Now, we review the definition of topological entropy given in [1]. Let X be a compact topological space and $f: X \rightarrow X$ a continuous map. For any two open covers \mathcal{Q} and \mathcal{B} of X let $\mathcal{Q} \vee \mathcal{B}$ denote $\{A \cap B: A \in \mathcal{Q} \text{ and } B \in \mathcal{B}\}$, and let $f^{-1}(\mathcal{Q})$ denote $\{f^{-1}(A): A \in \mathcal{Q}\}$. Let $M_n(f, \mathcal{Q})$ denote the minimum cardinality of a subcover of X of $\mathcal{Q} \vee f^{-1}(\mathcal{Q}) \vee \dots \vee f^{-(n-1)}(\mathcal{Q})$. We set

$$\text{ent}(f, \mathcal{Q}) = \lim_{n \rightarrow \infty} \frac{1}{n} (\ln(M_n(f, \mathcal{Q})))$$

where \ln denotes the natural logarithm. It is easy to see that this limit exists and is finite (see [1]). Finally we define the topological entropy of f by

$$\text{ent}(f) = \sup(\text{ent}(f, \mathcal{Q}))$$

where the supremum is taken over all open covers \mathcal{Q} of X . If X is a metric space it suffices to consider any sequence of open covers whose diameter approaches zero (see [1]). By the diameter of an open cover \mathcal{Q} we mean the supremum of the diameters of the open sets in \mathcal{Q} .

Two of the basic properties of topological entropy are the following (see [1]).

PROPOSITION 1. *Let X and Y be compact topological spaces, and let $f: X \rightarrow X$ and $g: Y \rightarrow Y$ be continuous maps. Suppose there is a continuous onto map $h: Y \rightarrow X$ such that $f(h(y)) = h(g(y))$ for all $y \in Y$. Then $\text{ent}(f) \leq \text{ent}(g)$. If h is a homeomorphism (in which case f and g are said to be topologically conjugate) then $\text{ent}(f) = \text{ent}(g)$.*

PROPOSITION 2. *If $f: X \rightarrow X$ is a continuous map of a compact topological space X , and K is a closed subset of X with $f(K) \subset K$, then $\text{ent}(f) \geq \text{ent}(f|_K)$.*

Let $f: X \rightarrow X$ be continuous. A point $x \in X$ is said to be wandering if there is a neighborhood V of x with $f^n(V) \cap V = \emptyset \ \forall n > 0$. The set of points which are not wandering is called the nonwandering set and denoted $\Omega(f)$. We state the following proposition from [4].

PROPOSITION 3. *Let $f: X \rightarrow X$ be a continuous map of a compact metric space. Then $\text{ent}(f) = \text{ent}(f|_{\Omega(f)})$.*

We next review some terminology concerning subshifts of finite type. Let A be an n by n matrix with all entries zero or one. Let X_A denote the set of infinite sequences $(.i_0i_1i_2 \dots)$ such that for each nonnegative integer j , $i_j \in \{1, 2, \dots, n\}$ and $A(i_j, i_{j+1}) = 1$, where $A(i_j, i_{j+1})$ denotes the entry of A in row i_j and column i_{j+1} . We have a map $s_A: X_A \rightarrow X_A$ given by

$$s_A(.i_0i_1i_2 \dots) = (.i_1i_2i_3 \dots).$$

This map is called the one-sided subshift of finite type associated to A . (If A is the n by n matrix with all entries one, s_A is called the full shift on n symbols.)

Note that X_A is a compact topological space and $s_A: X_A \rightarrow X_A$ is continuous. The topology on X_A is induced by the metric

$$d((.i_0i_1i_2 \dots), (.j_0j_1j_2 \dots)) = \sum_{k=0}^{\infty} 2^{-k}(\delta(i_k, j_k))$$

where $\delta(i_k, j_k) = 1$ if $i_k \neq j_k$ and $\delta(i_k, j_k) = 0$ if $i_k = j_k$.

We conclude this section by stating the following well-known proposition (see [4]).

PROPOSITION 4. *For any subshift of finite type $s_A: X_A \rightarrow X_A$, $\text{ent}(s_A) = \ln(\lambda)$, where λ is the largest eigenvalue of A .*

3. Proof of Theorem C.

LEMMA 5. *Let $f \in C^0(S^1, S^1)$. Let K_1, \dots, K_n be proper closed intervals of S^1 , such that for each $i = 1, \dots, n-1$, $f|_{K_i}$ is a homeomorphism and $f(K_i) \subset K_{i+1}$. Let \mathcal{Q} be a covering of $K_1 \cup \dots \cup K_n$ by finitely many open intervals, such that $\forall A \in \mathcal{Q}$ and $i = 1, \dots, n$, $A \cap K_i$ is either an interval or empty. Let $\text{card}(\mathcal{Q}) = k$ (card denotes cardinality). Then there is a subset \mathcal{B} of $\mathcal{Q} \vee f^{-1}(\mathcal{Q}) \vee \dots \vee f^{(-n+1)}(\mathcal{Q})$ which covers K_1 , with $\text{card}(\mathcal{B}) \leq n \cdot k$.*

PROOF. For each $i = 1, \dots, n$ choose a subset \mathcal{Q}_i of \mathcal{Q} which is a cover of K_i of minimum cardinality. Let $\mathcal{Q}_i = \{A_i(1), A_i(2), \dots, A_i(r_i)\}$ for $i = 1, \dots, n$. Let $K_i(j) = A_i(j) \cap K_i$ for $i = 1, \dots, n$ and $j = 1, \dots, r_i$, and let $\mathcal{C}_i = \{K_i(1), K_i(2), \dots, K_i(r_i)\}$ for $i = 1, \dots, n$.

Let $K_i = [a_i, b_i]$ for $i = 1, \dots, n$. For x and y in K_i let $d_i(x, y)$ denote the length of the arc joining x and y which is in K_i . If Y is a subset of K_i and $x \in K_i$, let $d_i(x, Y)$ denote $\inf\{d_i(x, y): y \in Y\}$.

By renumbering the elements of each \mathcal{C}_i (and each \mathcal{Q}_i) we may assume that the following are true (using the minimality of \mathcal{Q}_i):

- (1) $0 = d_i(a_i, K_i(1)) < d_i(a_i, K_i(2)) < \dots < d_i(a_i, K_i(r_i))$,
- (2) for $p = 2, \dots, n$,

$$0 = d_p(a_p, K_p(1)) < d_p(a_p, K_p(2)) < \cdots < d_p(a_p, K_p(r_p))$$

if $f^{(p-1)}$ is orientation preserving on K_1 while

$$0 = d_p(b_p, K_p(1)) < d_p(b_p, K_p(2)) < \cdots < d_p(b_p, K_p(r_p))$$

if $f^{(p-1)}$ is orientation reversing on K_1 .

Let $B_i(j) = K_i(j) - (K_i(1) \cup \cdots \cup K_i(j-1))$ for $i = 1, \dots, n$ and $j = 1, \dots, r_i$. Then $B_i(1) \cup \cdots \cup B_i(r_i) = K_i$ and $B_i(j) \cap B_i(m) = \emptyset$ for $j \neq m$. Also it follows from (1) and (2) above that the following is true:

(3) Suppose $p < m$, $j_1 \leq j_3$, $f^{(m-p)}(B_p(j_1)) \cap B_m(j_2) \neq \emptyset$, and $f^{(m-p)}(B_p(j_3)) \cap B_m(j_4) \neq \emptyset$. Then $j_2 \leq j_4$.

Let (j_1, j_2, \dots, j_n) denote $B_1(j_1) \cap f^{-1}(B_2(j_2)) \cap \cdots \cap f^{(-n+1)}(B_n(j_n))$ where $1 \leq j_i \leq r_i$ for $i = 1, \dots, n$. We write $(j_1, j_2, \dots, j_n) < (i_1, i_2, \dots, i_n)$ if $j_m \leq i_m \forall m = 1, \dots, n$ while for some $k = 1, \dots, n$, $j_k < i_k$.

It follows from (3) above that if (j_1, j_2, \dots, j_n) and (i_1, i_2, \dots, i_n) are nonempty and distinct then either $(j_1, j_2, \dots, j_n) < (i_1, i_2, \dots, i_n)$ or $(i_1, i_2, \dots, i_n) < (j_1, j_2, \dots, j_n)$. Hence we may list the nonempty distinct sets of the form (j_1, j_2, \dots, j_n) in order from the smallest to the largest (using the order $<$). It follows from the definition of $<$ that we can have at most $r_1 + r_2 + \cdots + r_n$ sets on the list. Since $r_i \leq k$ for $i = 1, \dots, n$, we have $r_1 + r_2 + \cdots + r_n \leq n \cdot k$. Thus there are at most $n \cdot k$ nonempty distinct sets of the form $(j_1, j_2, \dots, j_n) = B_1(j_1) \cap f^{-1}(B_2(j_2)) \cap \cdots \cap f^{(-n+1)}(B_n(j_n))$.

Let \mathfrak{B} denote the collection of sets of the form $A_1(j_1) \cap f^{-1}(A_2(j_2)) \cap \cdots \cap f^{(-n+1)}(A_n(j_n))$ such that (j_1, j_2, \dots, j_n) is nonempty. Then \mathfrak{B} is a subset of $\mathcal{Q} \vee f^{-1}(\mathcal{Q}) \vee \cdots \vee f^{(-n+1)}(\mathcal{Q})$ which covers K_1 and $\text{card}(\mathfrak{B}) \leq n \cdot k$. Q.E.D.

THEOREM C. Let $f \in C^1(S^1, S^1)$. Suppose there is a subset W of S^1 , and a finite collection of proper closed intervals of S^1 , $\{I(1), I(2), \dots, I(p)\}$, which satisfy the following:

(1) Either $W = S^1$ or $W = S^1 - (O_1 \cup O_2 \cup \cdots \cup O_m)$ where O_i is a component of the stable manifold of a contracting periodic point c_i for $i = 1, \dots, m$.

(2) $W = I(1) \cup I(2) \cup \cdots \cup I(p)$.

(3) For $j = 1, \dots, p$, f maps $I(j)$ homeomorphically onto its image.

(4) For any $i = 1, \dots, p$ and $j = 1, \dots, p$, $f(I(i)) \cap I(j)$ is an interval.

Then $\text{ent}(f) \leq \ln(p)$.

PROOF. Let

$$\delta = \min\{l(S^1 - I(1)), \dots, l(S^1 - I(p))\}$$

where $l(S^1 - I(j))$ denotes the length of the interval $S^1 - I(j)$. Let \mathcal{Q} be

any finite cover of $I(1) \cup I(2) \cup \dots \cup I(p)$ by open intervals with the diameter of \mathcal{Q} less than δ . Let $k = \text{card}(\mathcal{Q})$.

Let $I(j_1 j_2, \dots, j_n) = I(j_1) \cap f^{-1}(I(j_2)) \cap \dots \cap f^{(-n+1)}(I(j_n))$ where $j_i \in \{1, 2, \dots, p\} \forall i = 1, \dots, n$. Then each nonempty $I(j_1 j_2, \dots, j_n)$ is a closed interval and f maps $I(j_1 j_2, \dots, j_n)$ homeomorphically into $I(j_2, \dots, j_n)$. For any fixed $I(j_1 j_2, \dots, j_n)$, by Lemma 5 (with $K_i = I(j_i, \dots, j_n)$), there is a subset of $\mathcal{Q} \vee f^{-1}(\mathcal{Q}) \vee \dots \vee f^{(-n+1)}(\mathcal{Q})$ which covers $I(j_1 j_2, \dots, j_n)$ of cardinality at most $n \cdot k$.

Let $X = S^1 - (\cup_{i=1}^m W^s(c_i))$. Then X is a compact set with $f(X) \subset X$ and $f^{-1}(X) \subset X$.

Let $\mathcal{Q}(X)$ be the open cover of X defined by $\mathcal{Q}(X) = \{A \cap X : A \in \mathcal{Q}\}$. The minimal number of open sets of $\mathcal{Q}(X) \vee f^{-1}(\mathcal{Q}(X)) \vee \dots \vee f^{(-n+1)}(\mathcal{Q}(X))$ needed to cover X is equal to the minimal number of open sets of $\mathcal{Q} \vee f^{-1}(\mathcal{Q}) \vee \dots \vee f^{(-n+1)}(\mathcal{Q})$ needed to cover X . Now X is contained in the union of all the $I(j_1 j_2, \dots, j_n)$. Also, there are at most p^n distinct intervals of the form $I(j_1 j_2, \dots, j_n)$ and each one of these may be covered by a subset of $\mathcal{Q} \vee f^{-1}(\mathcal{Q}) \vee \dots \vee f^{(-n+1)}(\mathcal{Q})$ of cardinality at most $n \cdot k$. Hence $M_n(f|X, \mathcal{Q}(X)) \leq p^n \cdot n \cdot k$. This implies that $\text{ent}(f|X, \mathcal{Q}(X)) \leq \ln(p)$.

Since the diameter of $\mathcal{Q}(X)$ may be taken to be arbitrarily small, we have $\text{ent}(f|X) \leq \ln(p)$. But X contains all nonwandering points of f except for the finite set $\{c_1, c_2, \dots, c_m\}$. Thus

$$\text{ent}(f) = \text{ent}(f|\Omega(f)) = \text{ent}(f|X) \leq \ln(p). \quad \text{Q.E.D.}$$

4. Proof of Theorem D.

DEFINITION 6. Let $f \in C^1(S^1, S^1)$ and suppose that f has a sink and all singularities of f are in the stable manifolds of sinks. Let A be an m by m matrix. We say A is a matrix associated to f if there is a collection $\{I_1, \dots, I_m\}$ of pairwise disjoint closed intervals (or points) on S^1 such that the following hold:

- (1) For each $j = 1, \dots, m$, $f(I_j) \neq S^1$.
- (2) Let $W = S^1 - (I_1 \cup \dots \cup I_m)$. Then each component of W is a component of the stable manifold of a sink.
- (3) $f(W) \subset W$.
- (4) W contains all singularities of f .
- (5) $A(i, j) = 1$ if $f(I_i) \cap I_j \neq \emptyset$ and $A(i, j) = 0$ otherwise. Here $A(i, j)$ denotes the entry of the matrix A in row i and column j .

Note that conditions (3) and (4) of the definition imply that if $f(I_i) \cap I_j \neq \emptyset$ then $f(I_i) \supset I_j$. Also (1) and (4) imply that $f|I_j$ is a homeomorphism for $j = 1, \dots, m$.

If $f \in C^1(S^1, S^1)$ and f has a sink, and all singularities of f are in the stable manifolds of sinks (and if S^1 is not the stable manifold of a sink), then there

is a matrix associated to f . To obtain a matrix A associated to f note that the set of singularities of f , being compact, must be contained in finitely many components of stable manifolds of sinks. Let W be the union of all components of stable manifolds of sinks which contain points in the orbit of a singularity. Then W consists of finitely many components of stable manifolds of sinks and conditions (2), (3), and (4) (of Definition 6) are satisfied. We may add finitely many components of stable manifolds of sinks to W (if necessary) so that if I_1, \dots, I_m are the components of $S^1 - W$, then condition (1) is satisfied along with conditions (2), (3), and (4). Finally we use condition (5) to define the matrix A .

In the next section we will see some examples of mappings of the circle with associated matrices.

We remark that if f is expanding on the intervals I_1, \dots, I_m and if $K(f) = \{x \in S^1: f^n(x) \in I_1 \cup \dots \cup I_m \forall n > 0\}$ then $f|K(f)$ is topologically conjugate to the one-sided subshift of finite type s_A (see [5]). It is known (see [4]) that $\text{ent}(s_A) = \ln(\lambda(A))$ where $\lambda(A)$ is the largest eigenvalue of A . Thus in this case (using Propositions 2 and 3) it follows that $\text{ent}(f) = \ln(\lambda(A))$. In the more general case where f may not be expanding on the intervals I_1, \dots, I_m , $f|K(f)$ need not be conjugate to the shift s_A . In Theorem D we show that it is still true that $\text{ent}(f) = \ln(\lambda(A))$.

We will use the following elementary linear algebra fact in the proof of Theorem D.

PROPOSITION 7. *Let A be an m by m matrix with nonnegative entries. Let $\lambda(A)$ be the largest eigenvalue of A and let S_n denote the sum of the entries of A^n . Then*

$$\lim_{n \rightarrow \infty} (S_n)^{(1/n)} = \lambda(A).$$

PROOF. Define a norm on the space of all m by m matrices by setting $\|B\| = \sum |B(i,j)|$ where the sum is taken over all entries of B . It is easy to see that this satisfies the properties of a norm. Let $s(A)$ denote the spectral radius of A .

Since the matrix A is nonnegative, $\|A^n\| = S_n$ and $s(A) = \lambda(A)$. The conclusion now follows from the well-known theorem of analysis

$$\lim_{n \rightarrow \infty} \|A^n\|^{(1/n)} = s(A). \quad \text{Q.E.D.}$$

THEOREM D. *Let $f \in C^1(S^1, S^1)$. Suppose that f has a sink and all singularities of f are in the stable manifolds of sinks. Let A be a matrix associated to f in the sense of Definition 6. Then $\text{ent}(f) = \ln(\lambda(A))$, where $\lambda(A)$ is the largest eigenvalue of A .*

PROOF. Let I_1, \dots, I_m be the closed intervals which determine the matrix A

in the sense of Definition 6. Let $K(f) = \{x \in S^1: f^n(x) \in (I_1 \cup \dots \cup I_m) \forall n > 0\}$. Then $\text{ent}(f) = \text{ent}(f|K(f))$ since $K(f)$ contains all but finitely many points of $\Omega(f)$.

Let \mathfrak{B} be the open cover of $K(f)$ given by $\mathfrak{B} = \{I_1 \cap K(f), \dots, I_m \cap K(f)\}$. We will show

$$\text{ent}(f|K(f)) = \text{ent}(f|K(f), \mathfrak{B}).$$

Note that $M_n(f|K(f), \mathfrak{B})$ is equal to the number of nonempty distinct sets of the form $I_{j_1} \cap f^{-1}(I_{j_2}) \cap \dots \cap f^{-(n+1)}(I_{j_n})$. This is true because any two distinct sets of this form are disjoint, and any nonempty set of this form must contain an element of $K(f)$.

Let \mathcal{Q} be any open cover of $K(f)$ with the property that for each $A \in \mathcal{Q}$ there is an open interval V such that $A = V \cap K(f)$ and $V \cap I_j$ is an interval for $j = 1, \dots, m$. Let $\text{card}(\mathcal{Q}) = k$. By using Lemma 5 as in the proof of Theorem C we obtain

$$M_n(f|K(f), \mathcal{Q}) \leq n \cdot k \cdot M_n(f|K(f), \mathfrak{B}).$$

Hence $\text{ent}(f|K(f), \mathcal{Q}) \leq \text{ent}(f|K(f), \mathfrak{B})$. Since the diameter of \mathcal{Q} may be arbitrarily small we have $\text{ent}(f|K(f)) = \text{ent}(f|K(f), \mathfrak{B})$.

It follows from condition (5) of Definition 6 that $M_n(f|K(f), \mathfrak{B})$ is the sum of the entries of the matrix A^n . Thus by Proposition 7, $\text{ent}(f|K(f), \mathfrak{B}) = \ln(\lambda(A))$. We have

$$\text{ent}(f) = \text{ent}(f|K(f)) = \text{ent}(f|K(f), \mathfrak{B}) = \ln(\lambda(A)). \quad \text{Q.E.D.}$$

5. Construction of the sequence (f_n) . In this section we construct a sequence (f_n) of maps in $C^2(S^1, S^1)$ with $\text{ent}(f_n) \rightarrow \ln(3)$ as $n \rightarrow \infty$.

Let f_n be any map in $C^2(S^1, S^1)$ which satisfies conditions (1)–(6) of the definition of f_0 in §1 and also the following condition. Let $c^0 = c$. For $i = 1, 2, \dots, n-1$, let c^{-i} denote the unique inverse image (under f_n) of $c^{-(i+1)}$ in (e, l) , and let c^{-n} denote the unique inverse image (under f_n) of $c^{-(n-1)}$ in (t, k) . Then we require f_n to satisfy the condition that $f_n(t) \in \text{sism}(c^{-n})$. (Recall $\text{sism}(c^{-n})$ denotes the component of $W^s(c)$ which contains c^{-n} .)

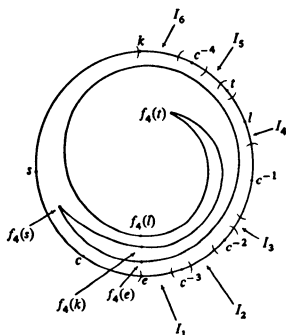


FIGURE 2

The map f_4 is pictured in Figure 2. In Figure 2, the open intervals indicated about $c^{-3}, c^{-2}, c^{-1}, t$, and c^{-4} represent the semilocal stable manifolds of these points.

Let I_1, I_2, I_3, I_4, I_5 , and I_6 be the components of the complement in S^1 of $\text{slsm}(c) \cup \text{slsm}(c^{-1}) \cup \text{slsm}(c^{-2}) \cup \text{slsm}(c^{-3}) \cup \text{slsm}(c^{-4}) \cup \text{slsm}(t)$ where we order these intervals as in Figure 2. Using these closed intervals we obtain a matrix A_4 associated to f_4 (see Definition 6 of §4) and

$$A_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $n \geq 4$, we may obtain a matrix A_n associated to f_n in the same manner as we did with f_4 , and A_n is the $n+2$ by $n+2$ matrix

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & & \\ \vdots & & & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the missing rows have ones on the super-diagonal and zeros elsewhere.

A straightforward calculation shows that

$$\det(A_n - \lambda I) = (\pm 1)(\lambda^{n+2} - 3\lambda^{n+1} + 2\lambda),$$

where I denotes the $n+2$ by $n+2$ identity matrix. Let a_n denote the largest root of this polynomial (the largest eigenvalue of A_n). Then $(a_n)^{n+1} - 3(a_n)^n + 2 = 0$. This implies $(a_n)^n(3 - a_n) = 2$. Since as $n \rightarrow \infty$, $(a_n)^n \rightarrow \infty$ it follows that as $n \rightarrow \infty$, $a_n \rightarrow 3$. But $\text{ent}(f_n) = \ln(a_n)$ by Theorem D, so that $\text{ent}(f_n) \rightarrow \ln(3)$ as $n \rightarrow \infty$. This is what we wanted to prove.

We state this result formally in the following.

PROPOSITION 8. *For the sequence (f_n) constructed above $\text{ent}(f_n) \rightarrow \ln(3)$ as $n \rightarrow \infty$.*

We next use the maps f_n to verify that (as noted in §1) entropy is not locally constant at f_0 .

PROPOSITION 9. *The map $\text{ent}: C^2(S^1, S^1) \rightarrow R$ is not locally constant at f_0 .*

PROOF. It is easy to see that arbitrarily close to f_0 (in $C^2(S^1, S^1)$) there is a map g_N which satisfies the same conditions as some map f_N in the sequence defined above. Thus $\text{ent}(g_N) = \ln(a_N)$ where a_N is the largest eigenvalue of A_N . From the proof of Proposition 8, we have $(a_N)^N(3 - a_N) = 2$, so $a_N < 3$. Hence $\text{ent}(g_N) < \ln(3)$. We will show in the proof of Theorem A (in §7) that $\text{ent}(f_0) = \ln(3)$ so this proves Proposition 9. Q.E.D.

6. A comparison theorem.

THEOREM 10. *Let $f \in C^1(S^1, S^1)$. Suppose f has a sink, and all singularities of f are in the stable manifolds of sinks (and suppose S^1 is not the stable manifold of a sink). Let A be a matrix associated to f and let I_1, \dots, I_m be the disjoint closed intervals which define the matrix A (see Definition 6). Let $g \in C^0(S^1, S^1)$ and suppose there are disjoint closed intervals J_1, \dots, J_m such that $f(J_j) \supset I_k$ implies that $g(J_j) \supset J_k$. Then $\text{ent}(g) \geq \text{ent}(f)$.*

PROOF. Let $s_A: X_A \rightarrow X_A$ denote the one-sided subshift of finite type associated to A (see §2 for definition). Let $K(g)$ denote the set of $x \in S^1$ that satisfy the following two conditions:

- (1) $g^k(x) \in J_1 \cup \dots \cup J_m$ for each nonnegative integer k .
- (2) If $g^k(x) \in J_{i_k}$ for each nonnegative integer k then $(.i_0 i_1 i_2 \dots) \in X_A$.

Note that $K(g)$ is a closed subset of S^1 . Define $h: K(g) \rightarrow X_A$ by $h(x) = (.i_0 i_1 i_2 \dots)$ where $g^k(x) \in J_{i_k}$ for each nonnegative integer k .

We show that h is onto. Suppose $(.i_0 i_1 i_2 \dots) \in X_A$. Then $f(I_{i_k}) \supset I_{i_{k+1}}$ for $k = 0, 1, 2, \dots$. Hence $g(J_{i_k}) \supset J_{i_{k+1}}$ for $k = 0, 1, 2, \dots$. It follows that there is at least one point $x \in S^1$ with $g^k(x) \in J_{i_k}$ for $k = 0, 1, 2, \dots$ (using the fact that a decreasing intersection of nonempty compact sets is nonempty). By definition $h(x) = (.i_0 i_1 i_2 \dots)$ so h is onto.

h is continuous because for any positive integer N and $x \in K(g)$, if $y \in K(g)$ is sufficiently close to x , then $g^k(x)$ and $g^k(y)$ will be in the same J_i for $0 \leq k \leq N$, so $h(x)$ and $h(y)$ will agree on the first N places. Clearly $s_A \circ h = h \circ g$, so by Propositions 1, 2, and 4 and Theorem D,

$$\text{ent}(g) \geq \text{ent}(g|K(g)) \geq \text{ent}(s_A) = \text{ent}(f). \quad \text{Q.E.D.}$$

7. Proof of Theorems A and B.

THEOREM A. *The map $\text{ent}: C^0(S^1, S^1) \rightarrow R \cup \{\infty\}$ is lower-semicontinuous at f_0 .*

PROOF. Let $\varepsilon > 0$. Pick N so large that $\ln(3) - \text{ent}(f_N) < \varepsilon$ (where f_N is the N th term of the sequence constructed in §5). Pick $\delta > 0$ such that if $d(f_0, g) < \delta$ (where d is a metric on $C^0(S^1, S^1)$) then the following seven conditions are satisfied.

(1) There are points $e(g)$, $c(g)$, $l(g)$, and $k(g)$ on S^1 corresponding to the points e, c, l , and k for f_0 , i.e. $e(g)$ is a fixed point of g near e , $c(g)$ is a fixed point of g near c , and $l(g)$ and $k(g)$ are points near l and k respectively with $g(l(g)) = g(k(g)) = e(g)$.

(2) g maps the interval $[e(g), l(g)]$ onto S^1 .

(3) There is a point $t(g) \in (l(g), k(g))$ such that for any $x \in [l(g), k(g)]$, $g(t(g)) \in [g(x), c(g)]$.

(4) There is a point $c^{-1}(g)$ in $(e(g), l(g))$ with $g(c^{-1}(g)) = c(g)$ and with the property that the image under g of the interval $(e(g), c^{-1}(g))$ contains the interval $(e(g), c(g))$, but is not equal to S^1 .

(5) $g(t(g)) \in (c^{-N}(g), c(g))$ where $c^{-N}(g)$ is any point of S^1 satisfying the following. For $i = 2, \dots, N-1$ let $c^{-i}(g)$ be a point in $(e(g), c^{(-i+1)}(g))$ with $g(c^{-i}(g)) = c^{(-i+1)}(g)$. Then let $c^{-N}(g)$ be a point in $(t(g), k(g))$ with $g(c^{-N}(g)) = c^{(-N+1)}(g)$.

(6) The image under g of the interval $(c^{-1}(g), t(g))$ contains the interval $(c(g), c^{-N}(g))$ but is not equal to S^1 .

(7) The image under g of the interval $(t(g), k(g))$ contains the interval $(e(g), c^{-N}(g))$ but is not equal to S^1 .

These conditions imply that the basic picture of g will be the same as that of f_0 in Figure 1, except that $g(t(g))$ need not be $k(g)$, and g may have many new "turning points".

Let I_1, I_2, \dots, I_{N+2} be the disjoint closed intervals which are the components of the complement in S^1 of $\text{slsm}(c) \cup \text{slsm}(c^{-1}) \cup \dots \cup \text{slsm}(c^{-N}) \cup \text{slsm}(t)$ where everything is defined for f_N as in §5, i.e. $c^0 = c$ and for $i = 1, \dots, N-1$, c^{-i} is the unique point in (e, l) with $f_N(c^{-i}) = c^{(-i+1)}$, and c^{-N} is the unique point in (t, k) with $f_N(c^{-N}) = c^{(-N+1)}$. Also we take I_1 to be the closed interval which contains e , I_2 the next one in the counter-clockwise direction, etc. (see Figure 2 for the case $N = 4$).

Let $g \in C^0(S^1, S^1)$ with $d(f_0, g) < \delta$. Let $O(c(g))$ denote the open interval $(k(g), e(g))$. Let $O(c^0(g)) = O(c(g))$. For $i = 1, \dots, N$ let $O(c^{-i}(g))$ denote the component of $f^{-1}(O(c^{(-i+1)}(g)))$ which contains $c^{-i}(g)$. Let $O(t(g))$ denote any open interval about $t(g)$ such that $g(O(t(g))) \subset (c^{-N}(g), c(g))$.

Let J_1, J_2, \dots, J_{N+2} be the disjoint closed intervals which are the components of the complement in S^1 of $O(c(g)) \cup O(c^{-1}(g)) \cup \dots \cup O(c^{-N}(g)) \cup O(t(g))$ where the J_i are ordered in the same manner as the I_i . Then $f_N(I_j) \supset I_k$ implies that $g(J_j) \supset J_k$ (see Figure 2 of §5 for the case $N = 4$), so $\text{ent}(g) \geq \text{ent}(f_N)$ by Theorem 10 of §6.

By the same argument as above it follows that $\text{ent}(f_0) \geq \text{ent}(f_n)$ for any integer $n > 2$. Hence by Proposition 8, $\text{ent}(f_0) \geq \ln(3)$.

By Theorem C of §3, $\text{ent}(f_0) \leq \ln(3)$. To apply Theorem C we take

$I(1) = [e, c^{-1}]$, $I(2) = [c^{-1}, t]$, and $I(3) = [t, k]$ (see Figure 1). Thus $\text{ent}(f_0) = \ln(3)$.

We have

$$\text{ent}(f_0) - \text{ent}(g) = \ln(3) - \text{ent}(g) \leq \ln(3) - \text{ent}(f_N) < \varepsilon. \quad \text{Q.E.D.}$$

THEOREM B. *The map $\text{ent}: C^2(S^1, S^1) \rightarrow R$ is continuous at f_0 .*

PROOF. Let d_2 denote a metric on $C^2(S^1, S^1)$. Let $\varepsilon > 0$. By Theorem A, $\exists \delta_1 > 0$ such that if $g \in C^2(S^1, S^1)$ with $d_2(g, f_0) < \delta_1$, then $\text{ent}(f_0) - \text{ent}(g) < \varepsilon$. By Theorem C, $\exists \delta_2 > 0$ such that if $g \in C^2(S^1, S^1)$ with $d_2(g, f_0) < \delta_2$ then $\text{ent}(g) \leq \ln(3)$. Here we are using the fact that for g sufficiently close to f_0 in $C^2(S^1, S^1)$, g will have exactly two singularities corresponding to those of f_0 , and we apply Theorem C as in the proof of Theorem A. Also, by the proof of Theorem A, $\text{ent}(f_0) = \ln(3)$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $d_2(g, f_0) < \delta$, we have $|\text{ent}(g) - \text{ent}(f_0)| = \text{ent}(f_0) - \text{ent}(g) < \varepsilon$. Q.E.D.

Finally, we remark that the entropy of a continuous map of the circle may be infinite (see §5 of [2] for an example), but the entropy of a C^1 map of the circle is finite (see Proposition 12 of [3]).

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